

# Topological Relations using Two Models of Uncertainty for Lines

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## Abstract

*This paper presents two models for determining the uncertainty in the topological relations between two lines. One model considers the uncertainty associated with the position of the line, while the other model captures the uncertainty associated with the position of the nodes. The first case considers a region of uncertainty surrounding the entire line and is called a broad line, whereas the second case considers two regions of uncertainty at the end points of the line and is called a line with a broad boundary. The 9-intersection as a generic model for binary topological relations identifies 33 different relations for lines without uncertainty. We found that for broad lines the number of distinguishable cases reduces to 5, while for lines with broad boundaries is extends to 77.*

Keywords: modeling uncertainty; lines; topological relations; broad lines; lines with broad boundaries.

## 1 Introduction

In the past spatial relations have been studied primarily for crisp spatial objects, such as points, lines, and regions. Popular models for such objects have been developed for topological relations (Egenhofer and Franzosa, 1991; Egenhofer and Herring, 1991), cardinal directions (Frank 1991, 1996; Goyal and Egenhofer 2000), and qualitative distances (Clementini *et al.* 1995; Hernández *et al.* 1996), each with a set of mutually exclusive and pairwise disjoint base relations that cover all possible configurations. When non-crisp spatial objects have been considered, typically either regions with broad boundaries (Clementini and Di Felice, 1996, 1997; Cohn and Gotts 1996), fuzzy regions (Tang and Kainz 2002, Winter 2000), or spatially-extended points (Lee and Flewelling 2004) have been used, but no systematic account has been made for relations with uncertain lines. The identification of a set of mutually exclusive and pairwise disjoint base relations is called the *problem of existence*. This paper addresses the problem of existence of topological relations between lines for two different uncertainty models.

Modelling the uncertainty of lines has an extensive history, such as epsilon bands (Chrisman 1982), concave curvilinear error bands (Dutton 1992), and G-bands (Shi and Liu 2000). These models all capture how the uncertainty in the location of a line's end nodes propagates along the line's interior. Since the different shapes that such error bands yield are topologically identical, we refer to them collectively as a *broad line*. Such a board line considers a region of

uncertainty surrounding the entire line; however, if the uncertainty is only associated with the position of the end nodes (i.e., the line's algebraic boundary), this yields two regions of uncertainty at the end nodes of the line. Such a model is called a *line with a broad boundary*. With a systematic study we determined the set of binary topological relations for these two different models. We found 5 possible topological relations between two broad lines, while there are 77 possible topological relations between two lines with broad boundaries.

The remainder of this paper is structured as follows: Section 2 summarizes background concepts used afterwards, namely the 9-intersections model for the representation of topological relations and the method to determine the existing topological relations. Sections 3 and 4 determine the existing topological relations between two broad lines and between two lines with broad boundaries, respectively. Section 5 discusses the results and offers conclusions.

## 2 Background

The 9-intersection (Egenhofer and Herring 1991) for the representation of topological relations between two objects  $A$  and  $B$  divides each object into three parts—the interior ( $A^o$ ), boundary ( $\partial A$ ) and exterior ( $A^-$ )—and determines topological invariants of the intersections of  $A$ 's and  $B$ 's parts. The most basic invariant is whether the intersections are empty ( $\emptyset$ ) or not empty ( $\neg\emptyset$ ). A 3x3 matrix  $I_n$ , called the 9-intersections matrix, records the nine intersections between  $A$ 's and  $B$ 's interiors, boundaries and exteriors (Equation 1). The 9-intersections matrix can also be represented iconically by a 3x3 grid, where black cells stand for non-empty intersections, whereas white cells stand for empty intersections.

$$I_n(A, B) = \begin{pmatrix} A^o \cap B^o & A^o \cap \partial B & A^o \cap B^- \\ \partial A \cap B^o & \partial A \cap \partial B & \partial A \cap B^- \\ A^- \cap B^o & A^- \cap \partial B & A^- \cap B^- \end{pmatrix} \quad (1)$$

The 9-intersections formalism is used as a basis for the study of the existence of topological relations. It distinguishes up to  $2^9=512$  different topological relations, but for most types of objects commonly used in geographic applications the number of possible relations is much smaller. For instance, eight of the 512 possible combinations can be realized between two regions (i.e., cells that are homeomorphic to 2-disks without any spikes or cuts). Between two simple lines (i.e., lines that have exactly two different end nodes, without any self-intersections or bifurcations), 33 of the 512 9-intersection matrices yield configurations that can be realized in  $\mathbb{R}^2$  (Egenhofer 1993). The boundary of a simple line is based on the algebraic boundary definition (Alexandroff 1961), its interior is the difference of the line's closure and its boundary, and the line's exterior is the complement of the line's closure.

The process used to determine the existing of topological relations has three steps (Egenhofer and Herring, 1991):

1. Creation of topological conditions, which are derived from the properties of the type of objects involved, and their translation into impossible patterns of the intersection matrix.
2. Determination of the possible topological relations by iteratively dropping from the set of 512 cases those that match the impossible patterns of the intersection matrices.

3. Finding for each matrix that remained in the previous step a corresponding geometric configuration whose topological relation fulfills all nine intersection constraints of that matrix.

### 3 Broad Lines

The uncertainty model of a broad line  $BL$  encompasses the sharp, simple line  $L$  such that  $BL$  contains  $L$  (i.e., all of  $L$  is fully contained in  $BL$ ). As such,  $BL$  does not retain the topological structure of a line but assumes that of a region. Unlike the definitions of error bands, which are based on metric properties, the broad-line definition is purely topological.

**Definition 1** - A broad line,  $BL$ , is a simple region representing a family of positions that a simple line can assume under continuous deformation.



Fig. 1. A broad line.

Such broad lines can be considered degenerate regions either with an empty boundary and a non-empty interior, or, as advocated by Clementini and di Felice (1997), with an empty interior and a non-empty (broad) boundary. In that case,  $\partial BL = BL$ , and the 9-intersection is reduced to the four intersections of  $BL$ s' boundaries and exteriors (Equation 2). In this paper, we adapt the latter definition.

$$I_n(A, B) = \begin{pmatrix} \partial A \cap \partial B & \partial A \cap B^- \\ A^- \cap \partial B & A^- \cap B^- \end{pmatrix} \quad (2)$$

For the four intersections, there are  $2^4=16$  possible intersection matrices (Figure 2), but not all of these combinations of empty and non-empty intersections correspond to cases that can be realized between two broad lines.

$\begin{pmatrix} \emptyset & \emptyset \\ \emptyset & \emptyset \end{pmatrix}$ BL1	$\begin{pmatrix} -\emptyset & \emptyset \\ \emptyset & \emptyset \end{pmatrix}$ BL2	$\begin{pmatrix} \emptyset & -\emptyset \\ \emptyset & \emptyset \end{pmatrix}$ BL3	$\begin{pmatrix} \emptyset & \emptyset \\ -\emptyset & \emptyset \end{pmatrix}$ BL4	$\begin{pmatrix} \emptyset & \emptyset \\ \emptyset & -\emptyset \end{pmatrix}$ BL5	$\begin{pmatrix} -\emptyset & -\emptyset \\ \emptyset & \emptyset \end{pmatrix}$ BL6
$\begin{pmatrix} -\emptyset & \emptyset \\ -\emptyset & \emptyset \end{pmatrix}$ BL7	$\begin{pmatrix} -\emptyset & \emptyset \\ \emptyset & -\emptyset \end{pmatrix}$ BL8	$\begin{pmatrix} -\emptyset & -\emptyset \\ -\emptyset & \emptyset \end{pmatrix}$ BL9	$\begin{pmatrix} -\emptyset & -\emptyset \\ \emptyset & -\emptyset \end{pmatrix}$ BL10	$\begin{pmatrix} -\emptyset & \emptyset \\ -\emptyset & -\emptyset \end{pmatrix}$ BL11	$\begin{pmatrix} -\emptyset & -\emptyset \\ -\emptyset & -\emptyset \end{pmatrix}$ BL12
	$\begin{pmatrix} \emptyset & -\emptyset \\ -\emptyset & \emptyset \end{pmatrix}$ BL13	$\begin{pmatrix} \emptyset & -\emptyset \\ \emptyset & -\emptyset \end{pmatrix}$ BL14	$\begin{pmatrix} \emptyset & -\emptyset \\ -\emptyset & -\emptyset \end{pmatrix}$ BL15	$\begin{pmatrix} \emptyset & \emptyset \\ -\emptyset & -\emptyset \end{pmatrix}$ BL16	

Fig. 2. The 16 combinations of intersections between boundaries and exteriors of broad lines.

In order to eliminate the impossible cases from these 16 combinations, we impose three conditions that must be met between two broad lines. In 9-intersection patterns of impossible relations the symbol “-” means that the value of the corresponding intersection can be either empty or non-empty.

**Condition 1:** The exteriors of two broad lines always intersect.

This means that  $A \cap B$  cannot be empty, therefore, the pattern for the impossible relations is

$R(A, B) \neq \begin{pmatrix} - & - \\ - & \emptyset \end{pmatrix}$ . This condition eliminates eight cases, namely BL1, BL2, BL3, BL4,

BL6, BL7, BL9 and BL13, leaving eight candidates.

**Condition 2:** The boundary of one broad line intersects at least one part of another broad line.

The impossible pattern is  $R(A, B) \neq \begin{pmatrix} \emptyset & \emptyset \\ \emptyset & - \end{pmatrix}$ . This condition is equivalent to assert that at

least one of the following three intersections must exist: boundary-boundary, boundary-exterior, or exterior-boundary. This condition eliminates BL5, such that seven candidates remain.

**Condition 3:** If the boundary of one broad line intersects the exterior of the other then either the boundaries intersect or the exterior of the first broad line intersects the boundary of the other, and vice-versa.

The impossible pattern is  $R(A, B) \neq \begin{pmatrix} \emptyset & \neg\emptyset \\ \emptyset & - \end{pmatrix} \vee \begin{pmatrix} \emptyset & \emptyset \\ \neg\emptyset & - \end{pmatrix}$ . This condition eliminates

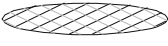
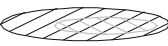
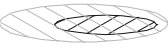

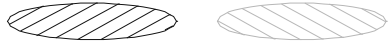
BL14 and BL16, leaving five candidates. For each of them one finds a geometric interpretation (Figure 3); therefore, the set of BL8, BL10, BL11, and BL12 forms the complete base set of mutually exclusive topological relations between two broad lines as it can be determined with the intersection method. These five configurations correspond to the set of five region-region relations that have been called the RCC-5 relations (Cohn and Gotts, 1996), which results from eliminating the distinctions made by empty or non-empty boundary-boundary intersections such that the region-region relations *contains* and *covers* are integrated into one single relation; *inside* and *coveredBy* are merged; and also *disjoint* and *meet* are combined.

#### 4 Lines with Broad Boundaries

Since the boundary of a simple line corresponds to the line's two end nodes, a broad boundary for a line means that nodes' get extended, while the line's interior, which connects the two nodes, remains unchanged (Figure 4).

**Definition 2** - A line with a broad boundary, *BBL*, is made up of a simple line, *LI*, and two simple regions, *d1* and *d2*, at the end points of the line (Clementini and Di Felice, 1997).

**Definition 3** - The broad boundary,  $\Delta BBL$ , of a line with a broad boundary, *BBL*, is the union of two simple regions *d1* and *d2*:  $\Delta BBL = d1 \cup d2$  (Clementini and Di Felice, 1997).

Relation	Intersection matrix	Topological relation	Geometric interpretation
BL8	$\begin{pmatrix} -\emptyset & \emptyset \\ \emptyset & -\emptyset \end{pmatrix}$	equal	
BL10	$\begin{pmatrix} -\emptyset & -\emptyset \\ \emptyset & -\emptyset \end{pmatrix}$	contains	
BL11	$\begin{pmatrix} -\emptyset & \emptyset \\ -\emptyset & -\emptyset \end{pmatrix}$	inside	
BL12	$\begin{pmatrix} -\emptyset & -\emptyset \\ -\emptyset & -\emptyset \end{pmatrix}$	overlap	
BL15	$\begin{pmatrix} \emptyset & -\emptyset \\ -\emptyset & -\emptyset \end{pmatrix}$	disjoint	

**Fig. 3.** Existing topological relations between two broad lines.



**Fig. 4.** A line with a broad boundary.

Boundary, interior, and exterior of a region with a broad boundary are still mutually exclusive, but their relations with respect to the crisp line show some important properties: (1)  $\Delta BBL$  contains  $\partial LI$  (i.e., one of  $LI$ 's nodes is contained in  $dI$  while the other node of  $LI$  is contained in  $L2$ ); (2)  $\Delta BBL$  intersects with  $LI^+$ ; (3)  $\Delta BBL$  intersects with  $LI^o$ ; (4)  $BBL^o$  is disconnected from  $\partial LI$ ;  $BBL^o$  is a true subset of  $LI$ 's interior; and (5)  $BBL$  is contained in  $LI^-$ .

The 9-intersection can be adapted to represent an object with a broad boundary by using the broad boundary,  $\Delta$ , instead of the sharp boundary (Equation 3).

$$R(A, B) = \begin{pmatrix} A^o \cap B^o & A^o \cap \Delta B & A^o \cap B^- \\ \Delta A \cap B^o & \Delta A \cap \Delta B & \Delta A \cap B^- \\ A^- \cap B^o & A^- \cap \Delta B & A^- \cap B^- \end{pmatrix} \quad (3)$$

To determine the set of realizable 9-intersection matrices for the relations between two lines with broad boundaries the same elimination process as used for determining the relations between two broad lines is used again.

**Condition 1:** The exteriors of two broad lines always intersect each other:

The impossible pattern is  $R(A, B) \neq \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & \emptyset \end{bmatrix}$ .

This constraint eliminates 256 cases from the 512 possible combinations, leaving 256 candidates.

**Condition 2:** The interior of  $A$  intersects at least one part (i.e., interior, broad boundary, or exterior) of  $B$ , and vice-versa.

The impossible pattern is  $R(A, B) \neq \begin{bmatrix} \emptyset & \emptyset & \emptyset \\ - & - & - \\ - & - & - \end{bmatrix} \vee \begin{bmatrix} \emptyset & - & - \\ \emptyset & - & - \\ \emptyset & - & - \end{bmatrix}$ . This constraint

eliminates another 56 cases, leaving 200 candidates.

The distinctive feature of a line with a broad boundary is that the boundary is composed of two simple regions. Thus, the boundary has an extent and cannot be equal to or contained in the interior of another line. This observation leads to the next two conditions for relations between lines with broad boundaries.

**Condition 3:** The boundary of  $A$  always intersects with  $B$ 's boundary or  $B$ 's exterior or both, and vice-versa.

The impossible pattern is  $R(A, B) \neq \begin{bmatrix} - & - & - \\ - & \emptyset & \emptyset \\ - & - & - \end{bmatrix} \vee \begin{bmatrix} - & - & - \\ - & \emptyset & - \\ - & \emptyset & - \end{bmatrix}$ . This condition eliminates

75 cases from the 200 that remained after applying conditions 1 and 2; therefore, 125 cases remain as candidates for realizable relations.

**Condition 4:** If the boundary of  $A$  intersects the interior of  $B$  then it must also intersect the exterior of  $B$ .

The impossible pattern is  $R(A, B) \neq \begin{bmatrix} - & - & - \\ -\emptyset & - & \emptyset \\ - & - & - \end{bmatrix} \vee \begin{bmatrix} - & -\emptyset & - \\ - & - & - \\ - & \emptyset & - \end{bmatrix}$ .

From the 125 candidates remaining after applying the first three conditions, another 48 are eliminated with this condition, leaving 77 candidates. Figures 5 and 6 show for each case a geometric example of two lines with broad boundaries whose topological relation fulfills all nine constraints, together with the corresponding 9-intersection matrix.

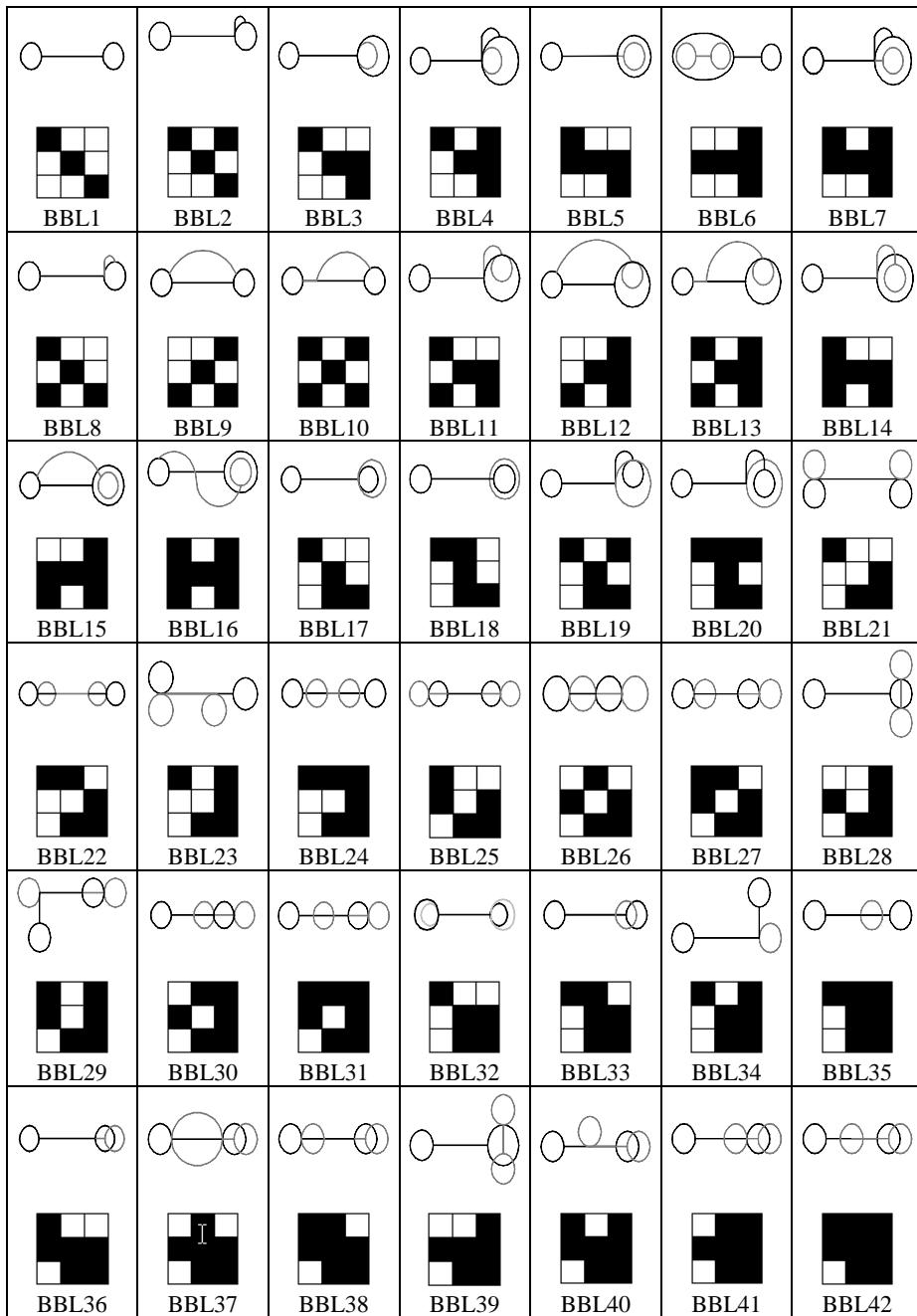
The set of conditions is not necessarily minimal (i.e., fewer than four conditions might be found to yield the same result), but no condition is part of another condition and no combination of the four conditions includes another condition. Also the sequence in which the conditions are applied has no influence on the resulting set of 77 matrices. The four conditions are necessary and sufficient to eliminate the cases impossible to be realized. They are

necessary because no condition is implied by another condition or by a combination of the other conditions, and they are sufficient because for each of the 77 9-intersection matrices a corresponding geometric configuration could be drawn.

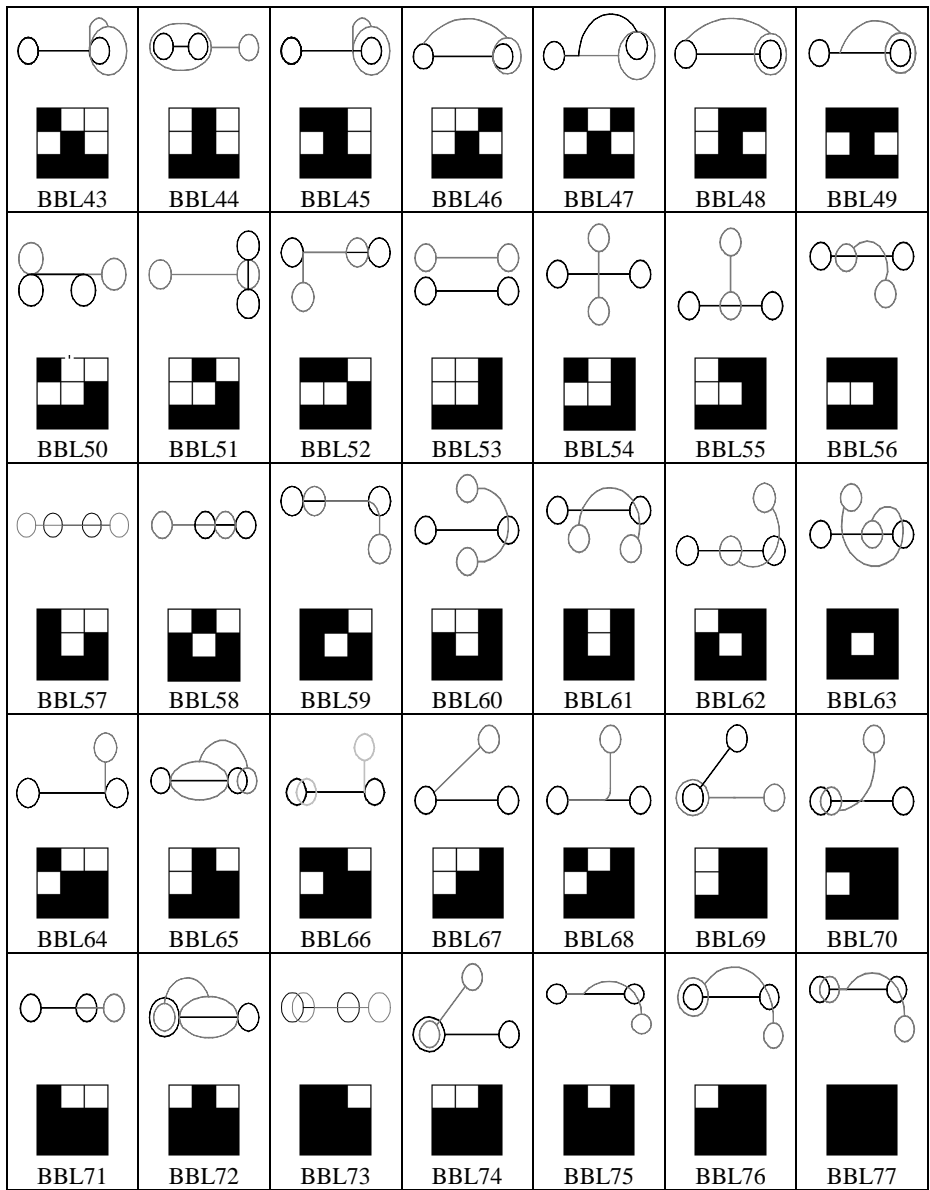
## 5 Conclusions

The problem of existence for topological relations between broad lines and lines with broad boundaries was studied by tailoring the 9-intersection to the two configurations and imposing conditions based on the specific properties of the objects involved. For broad lines the set of five topological relations was found that corresponds to the RCC-5 relations for region-region relations, whereas for lines with broad boundaries a total of 77 topological relations were found. The two abstract models are probably aimed at modeling two quite different scenarios of uncertainty: lines with broad boundaries capture uncertainty with respect to the lines' nodes and their connectivity; therefore, they are best suited for applications in graphs, such as subway diagrams. Broad lines, on the other hand, capture well uncertainty of lines that are determined from position measurements of their end nodes, as in the prototypical cadastral case.

The relations found with these two different models make a contribution to a better understanding of a spatial theory of spatial relations. From among the two models compared here, the model for lines with broad boundaries shows more an affinity to corresponding models for regions than the model for broad lines. The typical model for regions with broad boundaries (Cohn and Gotts, 1994, 1996) or undetermined boundaries (Clementini and Di Felice 1996) results in an increase of topological base relations from eight to 46, while the corresponding approach for topological relations between lines with broad boundaries increases the set of possibilities from 33 to 77. In absolute numbers the additional possibilities for broad-boundary objects are slightly higher for regions compared to line (38 vs. 34), but when compared to the cardinality of the crisp base set the increase in possibilities is far bigger for regions with broad boundaries than for lines with broad boundaries (5.75 times vs. 2.33 times as many possibilities as in the crisp scenario). Other comparisons can be made with other extensions of line-line relations. The increase for relations of lines with broad boundaries over crisp lines is the same order of magnitude as the increase for adding to crisp lines an orientation (Kurata and Egenhofer 2006) from 33 to 68, i.e., 35 more relations, or 2.06 times as many possibilities as in the case of lines without an orientation. On the other hand, the transitions from a linear to a cyclic embedding space for line relations (Hornsby *et al.* 1999) and for region relations (Egenhofer 2005) both showed only a modest increase in absolute and relative terms relations—from 13 relations to 16 relations (+23%) vs. from 8 to 11 relations (+38%). In this sense, the observed increase by capturing the uncertainty of lines with broad boundaries appears to display the largest expansion of possibilities when deviating from the prototypical crisp cases of relations in their native spaces.



**Fig. 5.** Forty-two out of 77 geometric interpretations and 9-intersection matrices of the topological relations between lines with broad boundaries



**Fig. 6.** Thirty-five out of 77 geometric interpretations and 9-intersection matrices of the topological relations between lines with broad boundaries

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